

Computational Hardness of Enumerating Satisfying Spin-Assignments in Triangulations

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Abstract

Satisfying spin-assignments in triangulations of a surface are states of minimum energy of the antiferromagnetic Ising model on triangulations which correspond (via geometric duality) to perfect matchings in cubic bridgeless graphs. In this work we show that it is NP-complete to decide whether or not a surface triangulation admits a satisfying spin-assignment, and that it is #P-complete to determine the number of such assignments. Both results are derived via an elaborate (and atypical) reduction that maps a Boolean formula in 3-conjunctive normal form into a triangulation of an orientable closed surface.

Keywords: Ising model; Triangulations; Groundstates; Parsimonious reduction; #P-complete.

1 Introduction

The Ising model is one of the most studied models in statistical physics. Characterizing its behavior on a system (graph) helps to understand physical phenomena associated to its thermodynamic properties [10]. The Ising model has been widely studied in lattices and regular structures (see for example [1, 7] and references therein). In contrast, irregular systems have received much less attention, probably due to the difficulty of deriving meaningful analytical results.

The number of distinct groundstates of the antiferromagnetic (negative coupling constant) Ising model of a system is called groundstate degeneracy and is typically exponentially large as a function of a parameter that measures the system's size (the number of nodes of the underlying graph). The latter translates to nonzero entropy at zero temperature when the system size goes to infinity, which in physical terms means that in the thermodynamical limit the spin arrangements of particles in the system is disordered. This partly explains the considerable attention

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physicist have given to developing techniques for approximating the groundstate degeneracy of a system.

Typically, researchers have focussed on developing techniques for bounding the groundstate degeneracy of a system, for example the Transfer Matrix Method [6, §6.6]. Instead, informally speaking, in this work we focus on the following two associated computational complexity problems; (1) hardness of deciding whether or not a given system admits a satisfying state, and (2) hardness of enumerating groundstates (equivalently, computing the groundstate degeneracy) of a given system. We show that the former problem is NP-complete and the latter is #P-complete.

We now make precise the notions discussed above and formally state our main results.

First, we describe the antiferromagnetic Ising model. We say that an embedding of a graph in an orientable closed surface is a *surface triangulation* if each face is bounded by a cycle of length 3 (in particular there is no loop) — multiple edges allowed. Given a triangulation T , let $V(T)$ and $E(T)$ denote the node and edge set of T . A mapping $s : V(T) \rightarrow \{-1, +1\}$ will be called a *spin-assignment* (state) to T . We refer to -1 and $+1$ as *spins*. The *energy* of a spin-assignment s of the antiferromagnetic Ising model is defined as $\sum_{uv \in E(T)} \sigma(u) \cdot \sigma(v)$. A *groundstate* is a spin-assignment of minimum energy. The number of distinct groundstates that a triangulation T admits is often referred to as the *groundstate degeneracy* of T . Clearly, under any spin-assignment to T the ends of at least one edge of each face of a surface triangulation T are both assigned either -1 or $+1$. Moreover, a spin-assignment is a groundstate if it has the smallest possible number of edges with both its ends being assigned the same spin. A face Δ of a surface triangulation T is said to be *frustrated* under assignment s , if s restricted to $V(\Delta)$ is not identically -1 or $+1$. A spin-assignment s to T is said to be *satisfying* (or *feasible*) if every face of T is frustrated under s . Obviously a satisfying spin-assignment is a groundstate. The converse is true for triangulations that can be embedded in the plane [5]. Nevertheless, the equivalence does not hold in general (the reader can verify that the toroidal triangulation depicted in Figure 1 does not have satisfying spin-assignments). However, note that when satisfying spin-assignments exist, then a groundstate is necessarily a satisfying spin-assignment.

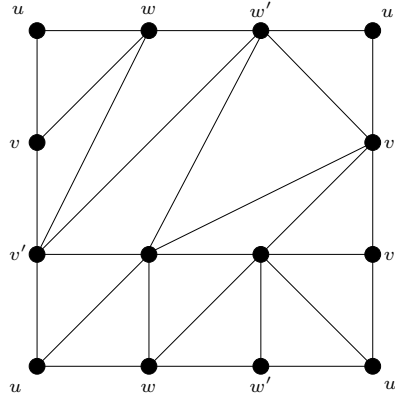


Figure 1: A triangulation of the torus with no satisfying spin-assignment.

In [5] a relation was established between the groundstate degeneracy of the antiferromagnetic and the number of perfect matchings in cubic bridgeless graphs. Specifically, let T be triangulation of a orientable closed surface and T^* its geometric dual. In [5], it is shown that the set of edges whose ends are assigned the same spin under a given satisfying spin-assignment to T correspond to a perfect matching of the cubic bridgeless graph T^* . Moreover, it is shown that if T admits a satisfying spin-assignment, then the groundstate degeneracy of T is at most twice the number of perfect matchings of T^* . Thus, lower bounds on the groundstate degeneracy of T provide lower bounds on the number distinct of perfect matchings of the cubic bridgeless graph T^* . An old and famous conjecture of Lovász and Plummer, recently positively settled [4], claimed that the number of distinct perfect matchings of T^* is exponential in the size (number of nodes) of T^* . The relation between number of satisfying spin-assignments of a surface triangulation and the number of perfect matchings of bridgeless cubic graphs is another one of our motivations for considering the problem of computing the groundstate degeneracy of surface triangulations.

In this work we show that the problem of deciding if a triangulation admits a satisfying spin-assignment is NP-complete. We also establish that computing the groundstate degeneracy of surface triangulations that admit satisfiable spin-assignments is #P-complete.

1.1 Contributions

Let T be a surface triangulation. For each v in $V(T)$ let $\delta(v)$ denote the set of edges incident to v . The map $\pi_v : \delta(v) \rightarrow \delta(v)$ is called *cyclic permutation of the edges incident to v* if for every e incident to v the edge $\pi_v(e)$ is the successor of e in the clockwise ordering around v defined by the surface embedding of T . The tuple $\pi = (\pi_v : v \in V(T))$ is called the *rotation system* of T . A direct consequence of the Heffter-Edmonds-Ringel rotation principle, is that every surface triangulation is uniquely determined, up to homeomorphism, by its rotation system [8, §3.2].

Let SatAssign be the collection of (encodings of) rotation systems of surface triangulations that admit a satisfying spin-assignment. Also, let #SatAssign be the function mapping (encodings of) rotation systems of surface triangulations to its number of satisfying spin-assignments.

To see that SatAssign is in NP, first recall that in order to check that $\pi = (\pi_v : v \in V(T))$ is an instance of SatAssign we need not start with a surface. Indeed, it suffices to check for every $v \in V(T)$ that π_v is a cyclic permutation of $\delta(v) = \{uv : uv \text{ is in } \pi_u \text{'s domain}\}$, a task that can be performed in time quadratic in $|V(T)|$ time in the Random Access Model. Then, observe that a certificate of membership in SatAssign of a rotation system of a surface triangulation T is simply a spin-assignment $s : V(T) \rightarrow \{-1, +1\}$ and that verifying that such an assignment is satisfying amounts to checking that each face Δ of the surface triangulation T is frustrated under s (which can be checked in $O(1)$ time per face in the Random Access Model).

In this work we establish the following results.

Theorem 1 SatAssign is NP-complete.

Theorem 2 #SatAssign is #P-complete.

Both of the stated results follow from an elaborate weakly parsimonious reduction [3, Definition 2.27] that maps a Boolean function in 3-conjunctive normal form to a rotation system of a triangulation (equivalently, to a triangulation embedded on a surface). As far as we are aware, this seems to be an atypical reduction, whose underlying ideas (e.g. gadgets) might be of independent interest due to their potential usefulness in the study of the computational hardness of other related spin glass problems.

2 Reduction idea and gadgets

The two main results of this work follow from reductions from a well known variant of the standard not-all-equal 3-satisfiability (abbreviated NAE-3SAT) problem which is known to be NP-complete even in the absence of negated variables [9], a variant we denote Positive-NAE-3SAT. For completeness sake, we recall in Figure 2 the precise definition of Positive-NAE-3SAT. Moreover, the counting version of Positive-NAE-3SAT, namely $\#$ Positive-NAE-3SAT, is $\#$ P-complete [2]. See Figure 3 for the precise definition of $\#$ Positive-NAE-3SAT.

PROBLEM	Positive-NAE-3SAT
INPUT	A Boolean formula φ in 3-conjunctive normal form such that each of its clauses C_1, \dots, C_m has exactly three (all non-negated) literals.
OUTPUT	TRUE if there is a truth assignment to φ such that for each clause C_i not all of its variables are assigned the same truth value.

Figure 2: Positive-NAE-3SAT.

PROBLEM	$\#$ Positive-NAE-3SAT
INPUT	A Boolean formula φ in 3-conjunctive normal form such that each of its clauses C_1, \dots, C_m has exactly three (all non-negated) literals.
OUTPUT	A (binary encoding) of the number of distinct truth value assignments to φ such for each clause C_i not all of its variables are assigned the same truth value.

Figure 3: $\#$ Positive-NAE-3SAT.

The overall strategy we will follow in proving Theorems 1 and 2 is fairly standard, i.e. we design gadgets where truth values of variables are set (choice gadgets) and gadgets where the truth value of clauses are evaluated (clause gadgets). We need to “carry” truth values from choice gadgets to clause gadgets, and make as many copies of the truth values taken by a literal as times they appear in all clauses. To achieve this task we build so called replicator gadgets. However, the construction of the aforementioned gadgets is quite delicate and non-obvious. In general, the main aspects we take care of in the construction of each gadget are existence and uniqueness of satisfying spin-assignments. However, there are subtle issues that need to be properly handled when building and piecing together the different gadgets. Below, we describe

in separate sections each of the gadgets we will require for the reduction and establish that they satisfy certain properties. First, we introduce some additional terminology and conventions we will use throughout the remaining part of this work.

2.1 Preliminaries

Note that given a triangulation T , a spin-assignment s to T is satisfying if and only if $-s$ is also a satisfying spin-assignment to T . We shall refer to this fact as *duality*. We will repeatedly use it in order to reduce the number of cases that need to be analyzed in order to establish some of the claims we will make. If T admits exactly two satisfying spin-assignments s and $-s$, we say that s ($-s$ respectively) is unique up to duality.

A 3-cycle in a triangulation will be called *positive* for a spin-assignment if at least two of its vertices are assigned spin $+1$. Otherwise, it will be called *negative*. This concept will be referred to as the *sign* of a 3-cycle.

Henceforth, if s assigns the same spin to all nodes of a subgraph H of T (respectively all elements of $S \subseteq V(T)$), we say that H (respectively a subset S) is *monochromatic* under s . Similarly, we say that an edge is monochromatic (respectively non-monochromatic) under s if s assigns the same (respectively distinct) spins to both ends of the edge. Monochromatic and non-monochromatic faces are defined analogously depending on whether or not its circumscribing cycle is either monochromatic or non-monochromatic. An edge e in $E(T)$ will be called *serious* if and only if e is monochromatic under every satisfying spin-assignment to T .

The gadgets we build in this work are embedded graphs in orientable closed surfaces with some removed disks (with holes) so that each face is bounded by a 3-cycle and each hole is circumscribed by a 3-cycle. In other words, every gadget may be obtained from a triangulation by cutting along the boundary of some of its faces (triangles). Thence, every term defined for surface triangulations is naturally adapted to gadgets so they will be reformulated only in case it is needed.

Throughout this work, serious edges are depicted as thicker lines and surface holes are depicted as gray areas.

2.2 Choice gadget

In this section we describe a gadget (a triangulation of a surface) that we will associate to Boolean variables in such a way that satisfying spin-assignments can be unambiguously interpreted as truth assignments to the Boolean variables. A *choice gadget* is a triangulation as depicted in Figure 4 embedded in a toroidal surface with one hole. The cycle with node set $\{u, v, w\}$ circumscribing the removed triangle of the choice gadget L , henceforth denoted by \mathcal{C}_L , will be referred to as the *variable cycle* of L and the edge uw will be called the *fundamental edge* of L (see Figure 4).

Our reduction will associate to each variable x_i a choice gadget L_i . A satisfying spin-assignment will be interpreted as setting x_i to TRUE if \mathcal{C}_{L_i} ends up being monochromatic, and

FALSE otherwise. The key functionality that we will show a choice gadget provides is that it has a unique up to duality satisfying spin-assignment where the variable cycle is monochromatic (respectively, non-monochromatic). Furthermore, choice gadgets will also be used as auxiliary building blocks in the construction of another type of gadget we will soon encounter.

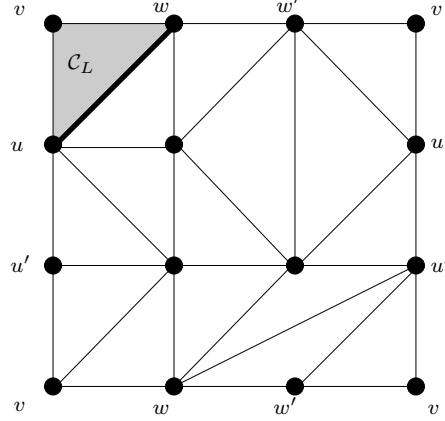


Figure 4: Choice gadget (region in gray depicts a surface hole).

The following result encapsulates the most relevant properties of choice gadgets.

Proposition 3 *Let L be a choice gadget. The fundamental edge of the choice gadget is serious. Moreover, there exists a unique up to duality feasible spin-assignment to L where the variable cycle \mathcal{C}_L of L is monochromatic (respectively, non-monochromatic).*

Proof: To prove the first statement, by duality, it suffices to show that there is no feasible spin-assignment extension to L when node v (node labels as in Figure 4) is assigned spin $+1$ and the fundamental edge uw is assigned spins $+-$ or $-+$. In Figure 5(a) we work out the case where uw is assigned spins $+-$; a subindex i accompanying a $+$ or $-$ sign indicates that the spin is forced by the spin-assignments with smaller indices in order for the assignment to be satisfiable — if spins assigned to the vertices of a triangle are forced to be all of the same sign, then no satisfying assignment can exist under the given initial conditions. The case when uw is assigned spins $-+$ is dealt with in the same way and worked out in Figure 5(b). This establishes that the fundamental edge of L is serious.

We now establish the claimed existence and uniqueness. Since the fundamental edge of L is serious, if s is a satisfying spin-assignment to L , then L 's fundamental edge is monochromatic under s . Therefore, again by duality, it is enough to prove that in the following two cases there exist exactly one feasible spin-assignment extension: (a) when the variable cycle \mathcal{C}_L (i.e. uvw) of L is assigned spin $+++$ (the monochromatic case), and (b) when it is assigned $++-$ (the non-monochromatic case). In Figure 6(a), the unique satisfying spin-assignment to L when nodes of its variable cycle are assigned $+1$ is exhibited.

For the non-monochromatic case, by duality and since L 's fundamental edge is serious, it suffices to consider the situation where L 's fundamental edge is assigned spins $++$. Two subcases arise,

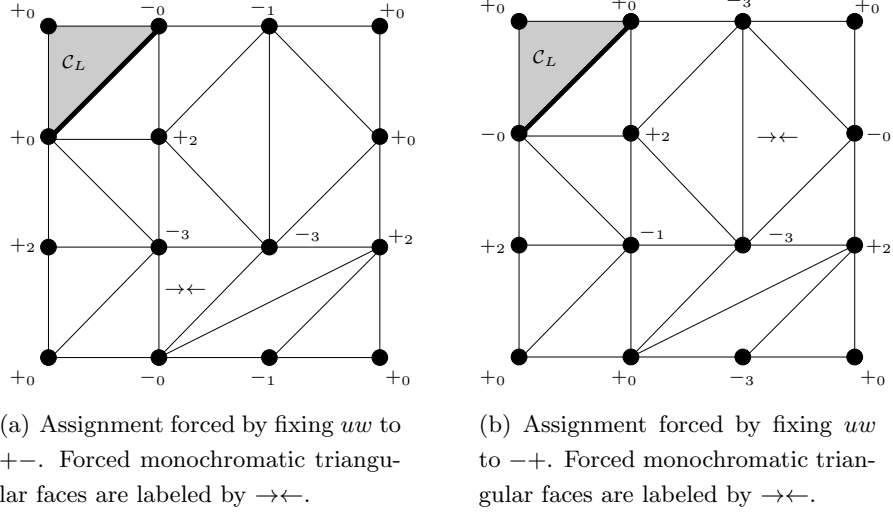


Figure 5:

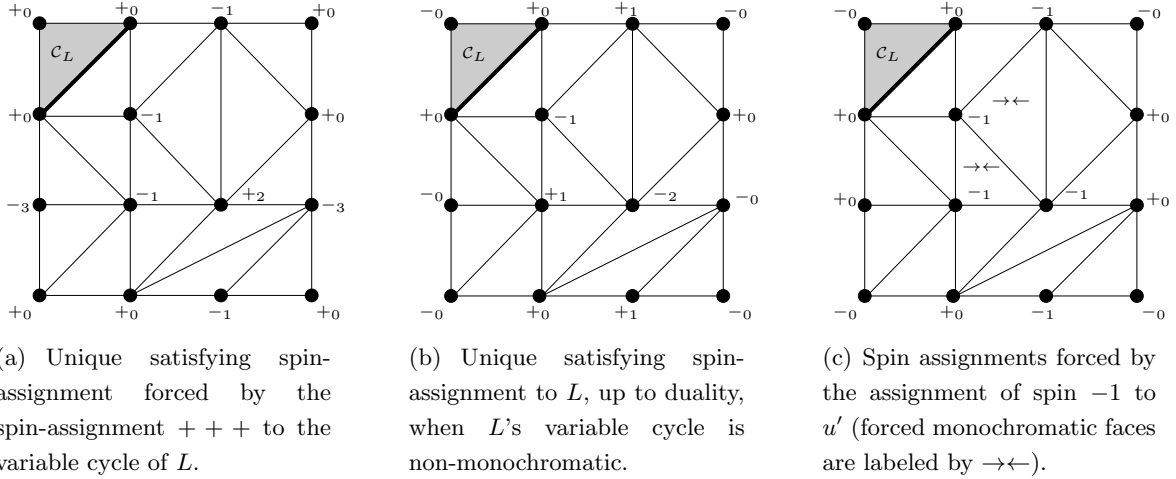


Figure 6:

depending on whether or not the spin $+1$ is assigned to node u' (node labels as in Figure 4) — each subcase is worked out separately in Figures 6(b) and 6(c). ■

2.3 Replicator gadget

A variable may appear several times in different clauses (or even multiple times in the same clause) of a Boolean formula in three conjunctive normal form. Given that the truth value a variable, say x_i , will be unambiguously set by the values taken by a satisfying spin-assignment at the associated choice gadget, say L_i , we require a way of “replicating” the encoding of the truth value of x_i as many times as x_i appears in the collection of formula clauses. To achieve this goal, to every choice gadget we will connect a special gadget, namely a k -replicator gadget. When the value k is clear from context or is not relevant, we will just write replicator gadget instead of

k -replicator gadget. For each $k > 0$, a k -replicator gadget will be a triangulation embedded in a surface of large genus (depending on k) with $2^k + 1$ holes. The purpose of a k -replicator gadget is to generate 2^k copies of the truth value encoded by a satisfying spin-assignment to the choice gadget to which the replicator gadget is connected.

To construct a k -replicator gadget we will glue together (in a particular way soon to be discussed) $2^k - 1$ so called block-replicator gadgets. A *block-replicator gadget* is a triangulation R of an orientable closed surface of genus 4 with three holes. A block-replicator gadget is built by gluing together three choice gadgets and the surface triangulation of the torus with six removed triangles depicted in Figure 7.¹ Specifically, the construction takes three choice gadgets, say \hat{L} , \bar{L} and \tilde{L} as depicted in Figure 4, and identifies the variable cycle $\mathcal{C}_{\hat{L}}$ (respectively, $\mathcal{C}_{\bar{L}}$ and $\mathcal{C}_{\tilde{L}}$) of the choice gadget \hat{L} (respectively, \bar{L} and \tilde{L}) with the cycle $\hat{\mathcal{C}} = u'xu$ (respectively, $\bar{\mathcal{C}} = xyw$ and $\tilde{\mathcal{C}} = zw'v$) of the block-replicator gadget R as depicted in Figure 7. The identification is done in such a way that edge $u'x$ (respectively, xy and zw') of the cycle $\hat{\mathcal{C}}$ (respectively, $\bar{\mathcal{C}}$ and $\tilde{\mathcal{C}}$) coincides with the fundamental edge of the choice gadget \hat{L} (respectively, \bar{L} and \tilde{L}). Clearly,

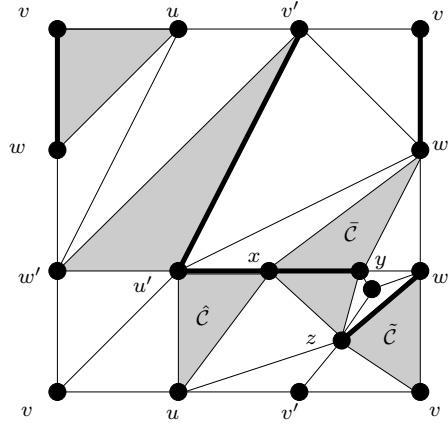


Figure 7: Block-replicator gadget.

under this construction, each block-replicator gadget has exactly three holes, because after gluing the choice gadgets to the surface triangulation depicted in Figure 7, the holes circumscribed by the cycles $\hat{\mathcal{C}}$, $\bar{\mathcal{C}}$ and $\tilde{\mathcal{C}}$ disappear. The length 3 cycle uvw circumscribing one of R 's hole is referred to as the *incoming cycle* (node labels as in Figure 7). The length 3 cycles circumscribing the other two holes of R will be called the *outgoing cycles*. Moreover, edges vw , $v'u'$ and xy will be referred to as *fundamental edges* of R (see again Figure 7 for node labeling scheme).

The attentive reader might wonder whether the described block-replicator gadget is indeed a surface triangulation. Specifically, whether indeed every “surface” point has an open neighborhood homeomorphic to some open subset of the Euclidean plane. This is indeed the case. Moreover, a block-replicator gadget has the following key property, henceforth referred to as *in-*

¹Here, choice gadgets are used as auxiliary gadgets. This auxiliary gadgets will not be associated to Boolean variables. The reason why we rely on this auxiliary choice gadgets is solely because of one of the properties we have shown they exhibit. Specifically, the fact that fundamental edges of choice gadgets are serious.

tersection property: the incoming and outgoing cycles of a block-replicator gadget do not share vertices. The intersection property implies that the surface on which the block-replicator gadget is embedded can be smoothly deformed into the one depicted in Figure 8.

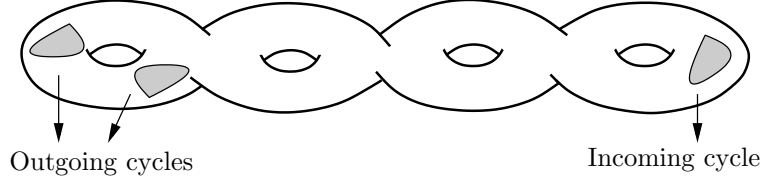


Figure 8: Block-replicator gadget sketch.

The key purpose of each block-replicator gadget is to enforce that if the incoming cycle is monochromatic (respectively, non-monochromatic), then both outgoing cycles will be monochromatic (respectively, non-monochromatic). Moreover, we will see that the block-replicator gadget inverts the sign of the incoming cycle, namely if the incoming cycle is positive (respectively, negative) the outgoing cycles are negative (respectively, positive). Actually, much more is true. Formally, we have the following results concerning block-replicator gadgets.

Proposition 4 *Let R be a block replicator gadget. Fundamental edges of R are serious. In particular, in every satisfying spin-assignment to R both outgoing cycles have the same sign and opposite to the sign of the incoming cycle.*

Proof: Given that xy is the fundamental edge of the variable cycle $\mathcal{C}_{\bar{L}} = xyw$ of the choice gadget \bar{L} , by Proposition 3, we have that xy is serious. To prove that vw is serious, by duality, it suffices to show that when vw is assigned $+-$ and u is assigned $+$ and $-$, there is no feasible spin-assignment extension to R . These two situations are worked out in Figure 9.

We proceed as above to prove now that $u'v'$ is serious. In Figure 10, we show that in the cases where $u'v'$ is assigned $-+$ and w' is assigned $+$ and $-$ there is no feasible spin-assignment extension to R . ■

Proposition 5 *Let R be a block-replicator gadget. Then, there exists a unique up to duality satisfying spin-assignments to R for which the incoming cycle is monochromatic (respectively, non-monochromatic). Moreover, if s is a satisfying spin-assignment to R , one of the two following statements hold:*

- (i). *the incoming and outgoing cycles are all monochromatic with the incoming cycle positive (respectively, negative) and both outgoing cycles negative (respectively, positive), or*
- (ii). *the incoming and outgoing cycles are all non-monochromatic with the incoming cycle positive (respectively, negative) and both outgoing cycles negative (respectively, positive).*

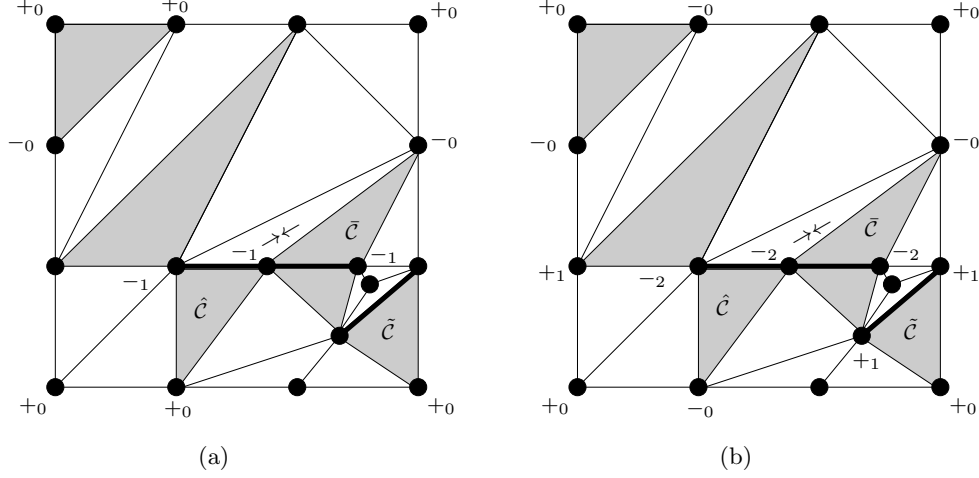


Figure 9: In (a), satisfying spin-assignment forced by fixing the outgoing cycle uvw' to $-++$ and in (b) to $-+-$. Serious edges are shown as thick lines. Note that fixing the spin of an end of a serious edge immediately forces the spin of its other end.

Proof: To prove existence and uniqueness of the satisfying spin-assignment when the incoming cycle is monochromatic, by duality, it is enough to prove that if all nodes in the incoming cycle uvw are assigned $+1$, then there exists a unique feasible spin-assignment extension to R (node labels as in Figure 7). This situation is worked out in Figure 11(a). On the other hand, by Proposition 4, if s is a satisfying spin-assignment to R , the edge vw belonging to the incoming cycle is monochromatic (because it is serious). Thus, by duality, we can assume that s assigns to the incoming cycle uvw spins $+++$ or $-++$. Therefore, to establish existence and uniqueness of the satisfying spin-assignment when the incoming cycle is non-monochromatic, it will suffice to show that there exists unique satisfying spin-assignment extension to R when the incoming cycle uvw is assigned spin $-++$. This case is studied in Figure 11(b). We need to check that each of the spin-assignments depicted in Figures 11(a) and 11(b) have a unique extension to the block-replicator gadget, even when the auxiliary choice gadgets are glued to the block-replicator gadget via proper identification of $\hat{C}, \bar{C}, \tilde{C}$ (see labels in Figure 8) and the variable cycles of the auxiliary choice gadgets. Proposition 3 and the fact that the spin-assignments depicted in Figures 11(a) and 11(b) completely determine the spins of the nodes of $\hat{C}, \bar{C}, \tilde{C}$ imply that the spin-assignment extensions to the whole block-replicator gadget are indeed feasible and unique. The remaining part of the claimed result can be ascertained by inspecting in Figures 11(a) and 11(b) the satisfying spin-assignments forced by the (non) monochromaticity of the incoming cycles. ■

We are now ready to describe the construction of a k -replicator gadget. Take $2^k - 1$ block-replicator gadgets $R_1, R_2, R_3, \dots, R_{2^k-1}$. Identify the outgoing cycles of R_1 with the incoming cycles of the block replicator gadgets R_2 and R_3 so that the fundamental edges of R_1 that belong to the outgoing cycles and the fundamental edges of R_2 and R_3 that belong to the

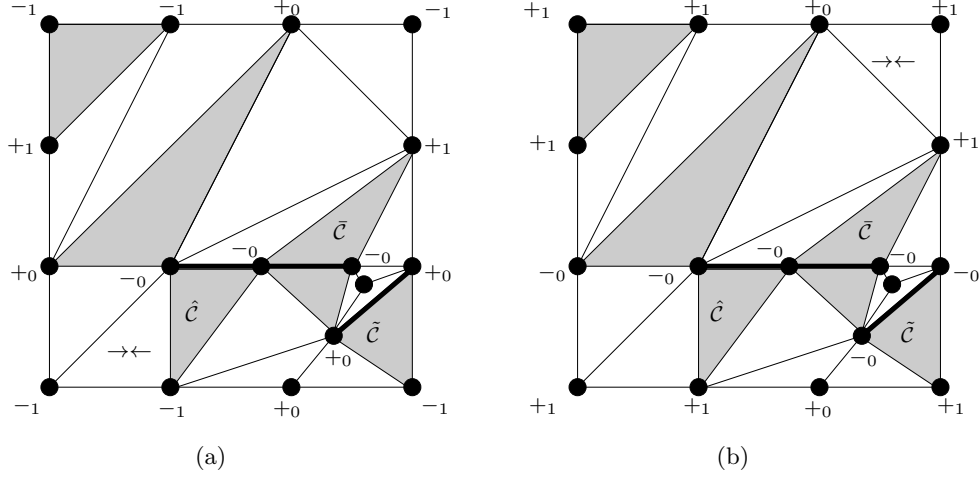


Figure 10: In (a), satisfying spin-assignment forced by fixing the outgoing cycle $u'v'w'$ to $- + +$ and in (b) to $- + -$. Serious edges are shown as thick lines. Note that fixing the spin of an end of a serious edge immediately forces the spin of its other end.

incoming cycles coincide. Continue in this way piecing together new block replicator gadgets and identifying fundamental edges, and construct a “rooted binary tree of depth k ” type structure of block-replicator gadgets. Let R_1 denote the block-replicator gadget at the “root” of the tree, and let $R_{2^{k-1}}, R_{2^{k-1}+1}, \dots, R_{2^k-1}$ denote the block-replicator gadgets at the “leaves” of the tree. The incoming cycle of R_1 will be referred to as the *starting cycle* of the k -replicator gadget R^k and the outgoing cycles of the block replicator gadgets $R_{2^{k-1}}, R_{2^{k-1}+1}, \dots, R_{2^k-1}$ will be called *end cycles* of R^k . Moreover, fundamental edges of the block-replicator gadget belonging to the incoming cycle of R_1 and to the outgoing cycles of $R_{2^{k-1}}, R_{2^{k-1}+1}, \dots, R_{2^k-1}$ will be referred to as fundamental edges of the k -replicator gadget.

Note that each k -replicator gadget is a triangulation of an orientable closed surface of genus $4 \cdot (2^k - 1)$ with $2^k + 1$ holes. Furthermore, the intersection property of the block-replicator gadgets is trivially transferred to k -replicator gadgets; namely, the starting and end cycles of a k -replicator gadget do not share vertices.

In our reduction, the starting cycle of each k -replicator gadget R^k will be identified with the variable cycle of a choice gadget, say L . By Proposition 5, this guarantees that the end cycles of R^k will be monochromatic if and only if the variable cycle of L is monochromatic. It is somewhat unfortunate that the block-replicator gadgets generate, at its outgoing cycles, encodings of opposite signs as the one of its incoming cycle. By taking k even, we can guarantee that each of the end cycles of R^k will have the same chromaticity (monochromatic or nonmonochromatic) and sign as the variable cycle of the choice gadget L . The following result captures all relevant properties we will need that are satisfied by replicator gadgets. The reader can easily check that the claimed properties are immediately inherited from those satisfied by block-replicator gadgets.

Corollary 6 *Let k be a positive integer and let R^k be a k -replicator gadget. The following*

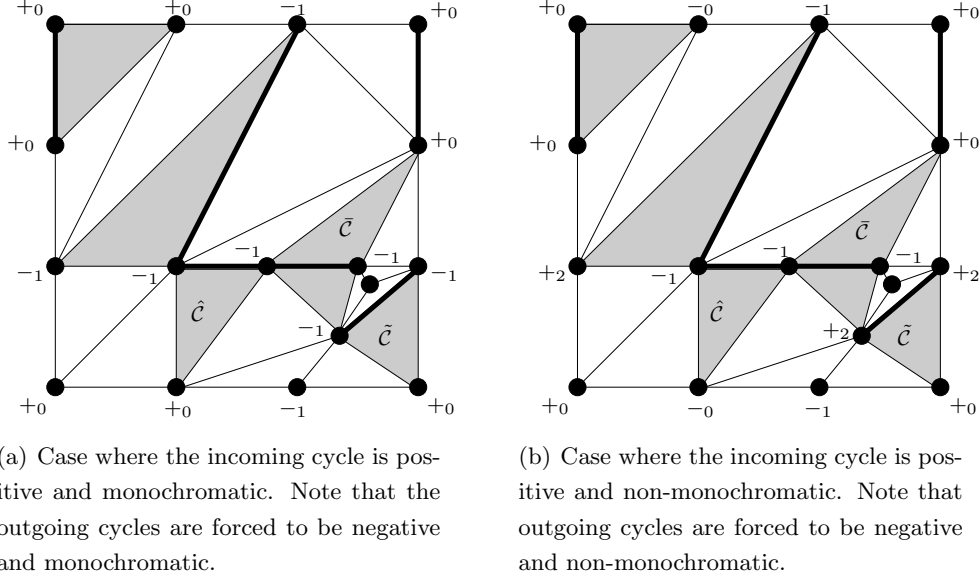


Figure 11:

statements hold:

- (i). *Fundamental edges of R^k are serious.*
- (ii). *For any satisfying spin-assignment to R^k , the starting cycle and the end cycles have the same sign.*
- (iii). *For any satisfying spin-assignment to R^k , the starting cycle and the end cycles are all either monochromatic or non-monochromatic.*
- (iv). *There is a unique up duality satisfying spin-assignment to R^k so that the starting cycle and the end cycles are all monochromatic (respectively, non-monochromatic).*

As we have already mentioned, the starting cycle of a replicator gadget, say R , will be identified with a variable cycle of a choice gadget, say L . Assuming that L is in turn associated to a formula variable, say x , it follows that in any satisfying spin-assignment all end cycles of R encode the same truth value of x encoded by the variable cycle of L . Eventually, some end cycles of R will be identified with cycles of the (next to be described) clause gadgets associated to formula clauses where x appears. If the total number of appearances of x in formula clauses is t , then R will be a k -replicator gadget where k is the smallest positive even integer greater or equal than $\log_2 t$. Thus, after identifying end cycles of R with cycles in clause gadgets, we might end up with non-identified end cycles (a situation that occurs whenever $\log_2 t$ is not a positive even integer). The holes circumscribed by such end cycles will need to be “capped” in order so at the end of our reduction we do indeed generate a surface triangulation. Moreover, holes will need to be “capped” in such a way that the properties satisfied by replicator gadgets are preserved. To achieve this goal, when necessary, we will identify an end cycle with the outer cycle of a *cap gadget* as depicted in Figure 12.

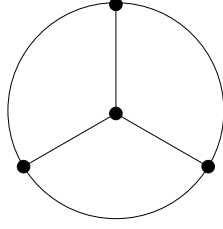


Figure 12: A cap.

The following statement is trivial.

Proposition 7 *For any spin-assignment to the outer cycle of a cap gadget, there exists a unique satisfying spin-assignment extension to the whole cap.*

2.4 Clause gadget

A *clause gadget* is a toroidal triangulation with three holes as depicted in Figure 13. The cycles circumscribing the holes of the clause gadget will be called *literal cycles*. Moreover, edges uw , $v'w$ and $v'u$ will be referred to as *fundamental edges* of the clause gadget (depicted as thicker lines in Figure 13).

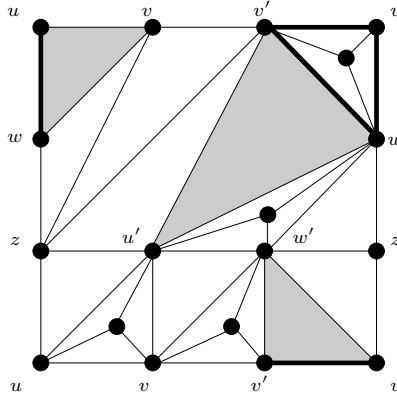


Figure 13: Clause gadget.

As already mentioned in the preceding section, we will eventually identify end cycles of replicator gadgets with literal cycles in such a way that fundamental edges of the replicator and clause gadgets coincide. In our reduction, replicator gadgets will “carry” from choice gadgets towards clause gadgets the encodings of the truth values of formula variables. The clause gadget is built in such a way as to allow a unique up to duality satisfying spin-assignment extension if and only if not all the truth value encodings “arriving” to the clause gadget represent the same truth value.

Unfortunately, fundamental edges of clause gadgets are not serious. However, once every literal cycle of a clause gadget is identified with an end cycle of a replicator gadget, fundamental

edges of the clause gadget will become serious in the triangulation thus formed — since fundamental edges of the replicator gadgets are serious, and because fundamental edges of the clause and replicator gadgets will be identified. This explains why when stating the following claims we assume seriousness of fundamental edges of the clause gadgets. The functionality provided by a clause gadget is summarized by the next results, the first of which is obvious.

Proposition 8 *Let C be a clause gadget. Assume that fundamental edges of C are serious. Then, for any satisfying spin-assignment, all literal cycles of C have the same sign.*

Proposition 9 *Let C be a clause gadget. Assume that fundamental edges of C are serious. In the following cases there is no satisfying spin-assignment extension to C :*

- (i). *when all literal cycles of C are monochromatic, and*
- (ii). *when all literal cycles of C are non-monochromatic.*

Proof: To prove the first claim, by duality and Proposition 8, it suffices to show that if all nodes in each literal cycle are assigned $+1$, then there is no feasible spin-assignment extension to C . This case is worked out in Figure 14(a).

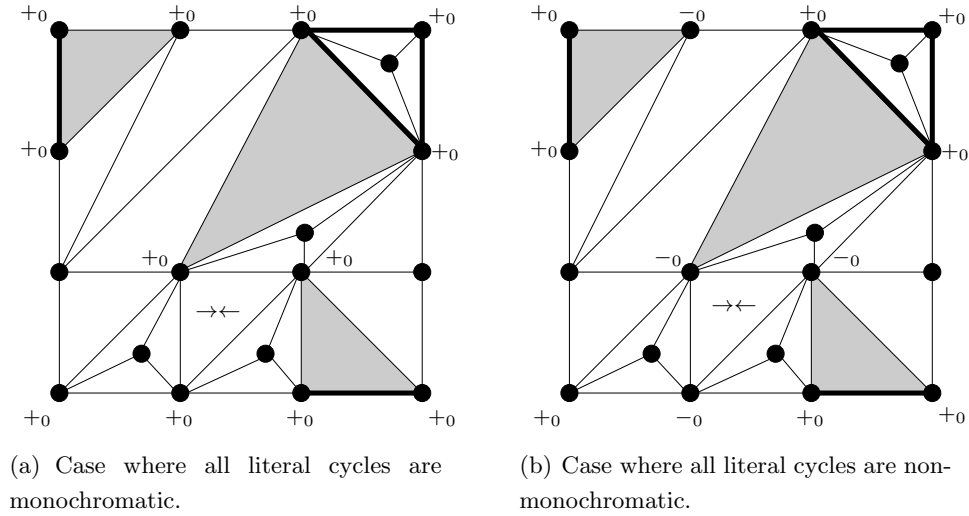


Figure 14:

To establish the second claim, it suffices to show that the same conclusion holds when the literal cycles uwv , $v'wu'$ and $v'u'w'$ are all assigned $++-$ (node labels as in Figure 13). This case is worked out in Figure 14(b). ■

Proposition 10 *Let C be a clause gadget. Assume that the fundamental edges of C are serious. If exactly one literal cycle of C is monochromatic (respectively, non-monochromatic) there is a unique up to duality satisfying spin-assignment extension to C .*

Proof: By duality and Proposition 8 the monochromatic case holds if we show that in each of the following situations there is exactly one feasible spin-assignment extension to C (node labels as in Figure 13): (a) when the literal cycle uvw is assigned $+++$ and spins $++-$ are assigned to cycles $v'wu'$ and $v'uw'$, (b) when the literal cycle $v'wu'$ is assigned $+++$ and spins $++-$ are assigned to cycles uvw and $v'uw'$, and (c) when the literal cycle $v'uw'$ is assigned $+++$ and spins $++-$ are assigned to cycles uvw and $v'wu'$. Each case is worked out separately in Figures 15(a), 15(b), and 15(c).

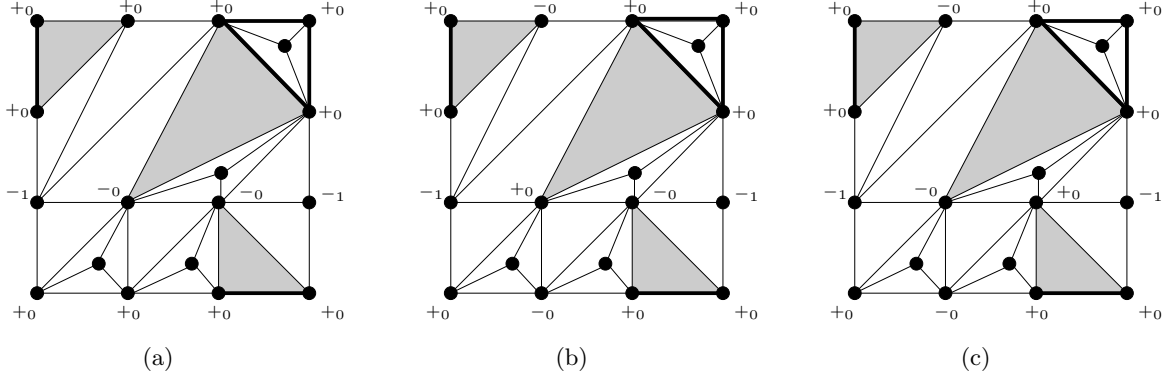


Figure 15: Unique forced satisfying spin-assignments to a clause gadget when exactly one literal cycle is monochromatic. Each shown spin-assignment encodes a truth value assignment to the variables of a clause where not all truth values are equal.

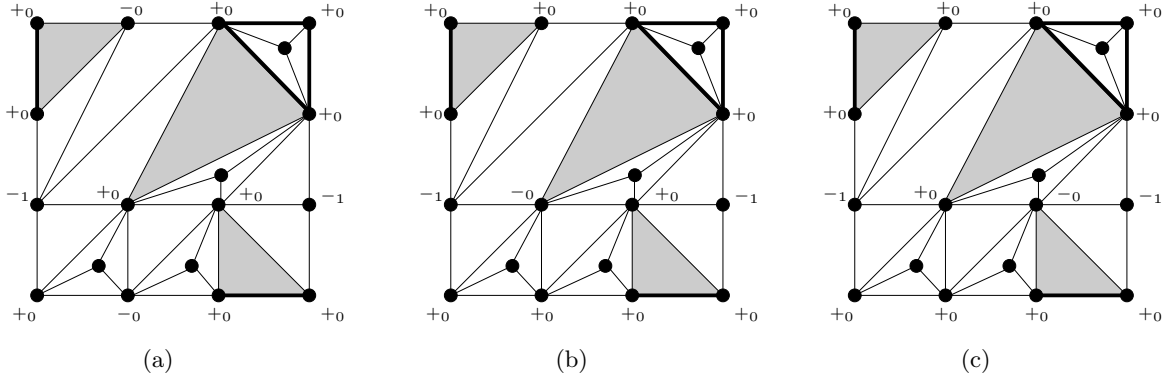


Figure 16: Unique forced satisfying spin-assignments to a clause gadget when exactly one literal cycle is non-monochromatic. Each shown spin-assignment encodes a truth value assignment to the variables of a clause where not all truth values are equal.

For the non-monochromatic case, we proceed in the same way. Again, By duality and Proposition 8, it suffices to examine the following situations (node labels as in Figure 13): (a) when the literal cycle uvw is assigned $++-$ and spins $+++$ are assigned to cycles $v'wu'$ and $v'uw'$, (b) when the literal cycle $v'wu'$ is assigned $++-$ and spins $+++$ are assigned to cycles uvw and $v'uw'$, and (c) when the literal cycle $v'uw'$ is assigned $++-$ and spins $+++$ are assigned to cycles uvw and $v'wu'$. Each case is worked out in Figures 16(a), 16(b), and 16(c). ■

3 The reduction

We now describe the reduction from Positive-NAE-3SAT to SatAssign. Let φ be a Boolean formula in conjunctive normal form, where each clause has exactly three (non-negated) literals. Let x_1, \dots, x_n be the variables and Cl_1, \dots, Cl_m be the clauses of φ . Let t_i denote the number of times variable x_i appears in the collection of clauses (multiple occurrences are counted multiple times). Define

$$k_i = \begin{cases} 2, & \text{if } t_i = 1, \\ 2\lceil (1/2) \log_2(t_i) \rceil, & \text{if } t_i > 1. \end{cases}$$

To each variable x_i we associate a choice gadget L_i . To each clause Cl_j we associate a clause gadget C_j . For $i = 1, \dots, n$, we identify the starting cycle of a k_i -replicator gadget R^{k_i} with the variable cycle of the choice gadget L_i so that the fundamental edge of L_i that belongs to L_i 's variable cycle and the fundamental edge of R^{k_i} that belongs to its starting cycle coincide. Note that the number of end cycles of R^{k_i} is at least t_i . For $i = 1, \dots, n$, identify t_i end cycles of R^{k_i} with literal cycles of the clause gadgets C_1, \dots, C_m where variable x_i appears in such a way that the fundamental edges of the end cycles of R^{k_i} and the fundamental edges of the literal cycles of the clause gadgets coincide. Identify the remaining $2^{k_i} - t_i$ end cycles of R^{k_i} (if any) with the outer cycle of a cap gadget. Denote by T_φ the surface triangulation thus obtained.

We first make a simple observation.

Lemma 11 *The fundamental edges of each clause gadget of T_φ are serious.*

Proof: Just observe that by construction of T_φ , fundamental edges of clause gadgets are identified with fundamental edges of replicator gadgets which are known to be serious, as established by Corollary 6 (i). ■

Note that, by construction, the surface on which T_φ is embedded is an orientable closed surface of genus $m + n + 4 \sum_{i=1}^n (2^{k_i} - 1)$ (1 due to each choice gadget L_1, \dots, L_n , another 1 due to each clause gadget C_1, \dots, C_m , and $4(2^{k_i} - 1)$ due to each replicator gadget R^{k_i} , $i = 1, \dots, n$). Clearly, since each 3-cycle circumscribing gadget holes were identified with a 3-cycle circumscribing another gadget hole, the surface on which T_φ is embedded *does not* have holes, i.e. it's a closed surface. Moreover, since the construction process does not create additional faces, each face of T_φ is a face of some gadget. Thence, each face is bounded by a 3-cycle, so T_φ is a triangulation. Finally, note that since each of the gadgets used in the construction of T_φ is embeddable in an orientable surface with holes, the resulting surface on which T_φ is embedded is also an orientable surface. Summarizing, T_φ is a triangulation of an orientable closed surface.

We say that φ is *connected* if for every non-trivial partition $\{S, \bar{S}\}$ of the clauses of φ (i.e. $S, \bar{S} \neq \emptyset$, $S \cap \bar{S} = \emptyset$, and $S \cup \bar{S}$ equals the set of clauses of φ) there is at least one variable that appears in one of the clauses in S and in one of the clauses of \bar{S} .

We now make a couple of useful observations.

Lemma 12 *Let φ be an instance of Positive-NAE-3SAT. If φ is connected, then the surface in which T_φ is embedded is also connected.*

Proof: Assume T_φ is embedded in a non-connected surface. Consider the set S of clauses whose associated clause gadgets are embedded in one of the connected surface components, say \mathcal{S} . Let \bar{S} be the collection of clauses not in S . Note that $\{S, \bar{S}\}$ is non-trivial. Moreover, the set of variables that appear in clauses in S (respectively, in \bar{S}) correspond to those variables associated to choice gadgets embedded in \mathcal{S} (respectively, not in \mathcal{S}). Both of these collection of variables must be disjoint, contradicting the fact that φ is connected. ■

Lemma 13 *Consider an instance φ of Positive-NAE-3SAT. Let Cl be a clause of φ and x_1, x_2 , and x_3 the (not necessarily distinct) variables appearing in Cl . Let L_1, L_2 , and L_3 be the (not necessarily distinct) choice gadgets of T_φ associated to x_1, x_2 and x_3 , respectively. Let C be the clause gadget of T_φ associated to Cl . Then, for every satisfying spin-assignment s to T_φ , the literal cycles of C and the variable cycles of L_1, L_2 , and L_3 have the same sign. Moreover, if φ is connected, then all literal cycles and variable cycles of T_φ have the same sign.*

Proof: By Proposition 8 and Lemma 11, under any satisfying spin-assignment to T_φ all literal cycles of C have the same sign. Without loss of generality, we can assume that all literal cycles of C are positive. Hence, the end cycles of the replicator gadgets, say R^{k_1}, R^{k_2} , and R^{k_3} , which are identified with the literal cycles of C , must all be positive. Since k_1, k_2 and k_3 are even, by Corollary 6 (ii), the starting cycles of R^{k_1}, R^{k_2} and R^{k_3} are positive. Given that the variable cycles of L_1, L_2 , and L_3 are identified in T_φ with the starting cycles of R^{k_1}, R^{k_2} and R^{k_3} , the first stated claim follows.

The last statement follows trivially from Lemma 12. ■

Theorem 14 *Let φ be an instance of Positive-NAE-3SAT. If φ is connected, then*

- (i). *For each truth value assignment that witnesses membership of φ in Positive-NAE-3SAT there is a unique up to duality satisfying spin-assignment to T_φ .*
- (ii). *For every pair of duality related satisfying spin-assignment to T_φ there is exactly one truth value assignment that witnesses membership of φ in Positive-NAE-3SAT.*

Proof: Let a_1, a_2, \dots, a_n be a truth value assignment to the variables x_1, \dots, x_n that is a witness of membership of φ in Positive-NAE-3SAT. We claim that there is a unique up to duality satisfying spin-assignment to T_φ . As usual, let Cl_1, \dots, Cl_m be the clauses of φ , let C_1, \dots, C_m denote the associated clause gadgets, and let L_1, \dots, L_n and R^{k_1}, \dots, R^{k_n} be the choice and replicator gadgets associated to variables x_1, \dots, x_n .

If a_i is TRUE, fix the spins of all nodes of the variable cycle of L_i in such a way that the cycle ends up being positive and monochromatic (observe that this can be done in a unique way). Otherwise, a_i is FALSE, fix the spins of all nodes of the variable cycle of L_i in such a way that the cycle ends up being positive and non-monochromatic (observe that since fundamental edges of choice gadgets are serious, this can again be done in a unique way). Extend the so far

partially defined satisfying spin-assignment to the union of choice gadgets (by Proposition 3, such a satisfying assignment extension exists and is unique).

Similarly, fix the spins of the nodes of the literal cycles of each clause gadget C_j according to the truth value taken by the associated Boolean formula variable, i.e. if the variable's value is TRUE, make the literal cycle positive and monochromatic, and positive and non-monochromatic otherwise (observe again that such spin-assignments can be done in a unique way). Extend once more the so far partially defined spin-assignment to the union of clause gadgets. We claim that such an extension exists and is unique. Indeed, given that a_1, \dots, a_n is a witness of membership of φ in Positive-NAE-3SAT, the variables in each clause Cl_j do not take the same truth value under the assignment a_1, \dots, a_n . Thus, the aforementioned spin-assignment to the literal cycles of C_j is such that not all literal cycles end up having the same chromaticity. By Proposition 10, each clause gadget has a unique satisfying spin assignment extension, thus establishing our claim.

Recall that variable cycles of choice gadgets (respectively, literal cycles of clause gadgets) are identified with starting cycles (respectively, end cycles) of replicator gadgets. Hence, for all $i = 1, \dots, n$, the partial spin-assignment thus far defined makes the start and end cycle of the replicator gadget R^{k_i} positive and monochromatic (respectively, non-monochromatic) if and only if a_i is TRUE (respectively, FALSE). Given that k_i is even and positive for all $i = 1, \dots, n$, by Corollary 6 (iv) and Proposition 7, there is a unique extension of the previously defined partial spin-assignment to all nodes of replicator gadgets so the resulting spin-assignment is a satisfying spin-assignment for T_φ .

We now prove the second part of the claimed result. Assume there is a satisfying spin-assignment s to T_φ . By Lemma 13 and since φ is connected, all literal and variable cycles of T_φ have the same sign, say positive. For $i = 1, \dots, n$, let a_i be TRUE if the variable cycle of L_i is positive and monochromatic, and FALSE if the variable cycle of L_i is positive and non-monochromatic. We claim that a_1, \dots, a_n is a witness of membership of φ in Positive-NAE-3SAT. Indeed, assume x_{j_1} , x_{j_2} , and x_{j_3} are the (not necessarily distinct) variables appearing in clause Cl_j . Let $s \in \{1, 2, 3\}$. Since the start cycle of the replicator gadgets $R^{k_{j_s}}$ is identified with the variable cycle of the choice gadget L_{j_s} , then by Corollary 6 the end cycles of $R^{k_{j_s}}$ must be monochromatic if and only if a_{j_s} is TRUE. Since the non-capped end cycles of replicator gadgets are identified with the literal cycles of clause gadgets, we have that the literal cycle of the clause gadget C_j associated to the variable x_{j_s} is monochromatic if and only if a_{j_s} is TRUE. Moreover, all literal cycles of clause gadgets are positive. Since s is a satisfying spin-assignment, by Proposition 9 and Proposition 10, the literal cycles of C_j can not all be either monochromatic or non-monochromatic. This implies that a_{j_1} , a_{j_2} , and a_{j_3} are not all equal, as we wanted to establish. ■

Corollary 15 *Let φ be an instance of Positive-NAE-3SAT. Then, φ is satisfiable if and only if there is a satisfying spin-assignment to T_φ .*

Proof: If φ is connected, the result is immediate from Theorem 14. Assume φ is not con-

nected. Then, there are $\varphi_1, \dots, \varphi_c$ instances of Positive-NAE-3SAT such that each φ_i is connected and $\varphi = \bigwedge_{i=1}^c \varphi_i$. Moreover, φ belongs to Positive-NAE-3SAT if and only if $\varphi_1, \dots, \varphi_c$ belong to Positive-NAE-3SAT. By Theorem 14, this is equivalent to saying that φ belongs to Positive-NAE-3SAT if and only if the union of the surface triangulations $T_{\varphi_1}, \dots, T_{\varphi_c}$, i.e. T_φ , admits a satisfying spin assignment. ■

The next result allows us to handle, in our reduction, instances of Positive-NAE-3SAT which are not connected.

Lemma 16 *Let φ be an instance of Positive-NAE-3SAT. Then, there exists a log-space (hence, polynomial time) computable instance φ' of Positive-NAE-3SAT such that φ' is connected and the following equality holds:*

$$|\{\vec{a} = (a_1, \dots, a_n) : \vec{a} \text{ is a witness of membership of } \varphi \text{ in Positive-NAE-3SAT}\}| = \frac{1}{2} |\{\vec{a}' = (a'_1, \dots, a'_n) : \vec{a}' \text{ is a witness of membership of } \varphi' \text{ in Positive-NAE-3SAT}\}|.$$

Proof: Assume x_1, \dots, x_n are the variables and C_1, \dots, C_m the clauses of φ . Consider two additional Boolean variables y and z and define n additional clauses C'_1, \dots, C'_n such that $C'_i = x_i \wedge y \wedge z$. Let φ' be the conjunction of $C_1, \dots, C_n, C'_1, \dots, C'_n$. Clearly, φ' is a connected instance of Positive-NAE-3SAT which is log-space computable given φ . Note that a membership witness a_1, \dots, a_s of an instance of Positive-NAE-3SAT can not be such that all a_i 's are equal. This immediately implies that x_1, \dots, x_n is a witness of membership of φ in Positive-NAE-3SAT if and only if $x_1, \dots, x_n, y = 0, z = 1$ and $x_1, \dots, x_n, y = 1, z = 0$ are witnesses of membership of φ' in Positive-NAE-3SAT. ■

Corollary 17 *Let φ be an instance of Positive-NAE-3SAT. Then, there exists a log-space (hence, polynomial time) computable instance φ' of Positive-NAE-3SAT such that*

$$|\{s : s \text{ is a satisfying spin-assignments to } T_{\varphi'}\}| = 4 \cdot |\{\vec{a} = (a_1, \dots, a_n) : \vec{a} \text{ is a witness of membership of } \varphi \text{ in Positive-NAE-3SAT}\}|.$$

To conclude, note that given an instance φ of Positive-NAE-3SAT with n variables and m clauses each of the necessary gadgets can be constructed in $O(\log |V(T'_\varphi)|)$ space, i.e. in log-space in the size of the encoding of φ . The number of choice, block-replicator, and clause gadgets that need to be built are $n + 2$, $\sum_{i=1}^n (2^{k_i} - 1) = O(m)$, and $m + n$, respectively. Hence, an encoding of a rotation system for $T_{\varphi'}$ can be computed in log-space, thence in polynomial time. Then, Corollary 15 and Corollary 17 imply Theorem 1 and Theorem 2, respectively.

To conclude, note that this work does not preclude that the problem of deciding whether or not a triangulation embedded in surfaces of a fixed constant genus admits a satisfying spin-assignment might be say decidable in polynomial time. Neither does it preclude that the associated counting problem might be significantly easier than $\#P$ -hard. We believe this are interesting issues to study.

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